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On the Initiation Problem for a Combustion Model

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The initiation problem for a combustion model is studied. Global existence of the solution to the problem is established and the asymptotic behavior of the solution is studied. Specifically, we prove that the solution converges to a self-sustaining detonation wave, i.e., the minimal travelling wave, or the CJ detonation wave, followed by a rarefaction wave provided that the initial pulse is large. If the data are small, the solution decays to zero like an *N-wave*.

1. INTRODUCTION

In this paper we address the initiation problem for the combustion model introduced by Rosales and Majda [9],

$$u_t + (\tfrac{1}{2}u^2 - q_0 z)_x = 0 \tag{1}$$

$$z_x = K\varphi(u) z^\alpha, \tag{2}$$

where q_0, $K > 0$ are constants and $\varphi(u)$ has the ignition form

$$\varphi(u) = \begin{cases} 1, & u \geqslant u_i, \\ 0, & u < u_i, \end{cases} \tag{3}$$

and $0 < u_i < \sqrt{2q_0}$. For their physical meaning, we refer the readers to [9].

As suggested by Roytburd [10], we take $0 < \alpha < 1$ in this paper. In this case the width l of the reaction zone is

$$l = \frac{1}{K(1-\alpha)},$$

which is finite so that there is a special solution with compact support and which travels at a constant speed [10].

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The system admits detonation wave solution. The leading part of a detonation front is a strong shock wave popagating into the explosive. The shock heats the material by compressing it, thus triggering chemical reaction and a balance is attained such that the chemical reaction supports the shock. Detonation waves are travelling wave solutions which have the above described structure. We note that the system has no travelling wave solution for detonation velocity below a certain minimum velocity [2], [9]. The solution with the minimum velocity is called Chapman–Jouguet or CJ detonation. The flow immediately behind the CJ detonation front is sonic with respect to the front. With the usual rear boundary condition such as a rigid wall, the only steady solution is a CJ detonation followed by a rarefaction wave. In this case, the head of the rarefaction wave moves at precisely the same speed as the front because of the sonic property of the CJ wave [4], [11].

The data we are considering are

$$u(x,0)=\begin{cases} u_0, & 0<x<d,\\ 0, & \text{elsewhere,}\end{cases} \tag{4}$$

$$z(+\infty,t)=1, \tag{5}$$

where $u_0>0$ and $d>0$.

The initiation problem is: When will the initial pulse develope into a self-sustaining detonation wave and when will it decay to zero as $t\to+\infty$?

We prove that the solution to (1), (2), (4), and (5) exists globally and converges to the travelling wave with the minimum speed followed by a rarefaction wave if the initial data d and u_0 are large and decays to zero like an *N-wave* [6] if the date are small.

This phenomena is previously observed in the numerical computations of Bourlioux [1]. This is the competing result of two main wave propagation mechanisms: one is wave attenuation caused by the nonlinear spreading of rarefaction waves; the other is wave amplification due to exothermic heat release [8].

It is interesting to note that analogous results, solutions with compact support initial data converge to minimum travelling waves, have also been obtained for certain reaction–diffusion equations in the celebrated paper [3] by Kolmogoroff *et al.*

In [5], we studied the Riemann problem of the above model where $\alpha=1$. All the conclusions there, namely, that solution of the Riemann problem exists globally and converges to travelling wave solution, also hold in the case $0<\alpha<1$. We will summarize the results about the Riemann problem in Section 2. Also see [12].

In Section 3, we take a special compact support function as the initial data: its support is $[0,l]$, within the support it is a travelling wave. In this case we can get concaveness of the shock front and hence get its asymptotic behavior. We prove that the solution converges to a CJ wave followed by a rarefaction as $t \to +\infty$.

In Section 4, we prove the convergence results for the initiation problem by the comparison principle and results in Section 3.

The tools used in our proof are the method of characteristics, a maximum principle, and the conservative property of (1).

2. The Riemann Problem

Let us first give travelling wave solutions $(\psi(x-Dt), z(x-Dt)) = (\psi(\xi), z(\xi))$ of (1) and (2), where $\xi = x - Dt$,

$$(\psi(\xi), z(\xi)) = \begin{cases} (0, 1), & \xi \geqslant 0, \\ (D + (D^2 - 2Du_0 + u_0^2 + 2q_0 z(\xi))^{1/2}, (K(1-\alpha)\xi + 1)^{1/(1-\alpha)}), & \\ \qquad -l \leqslant \xi < 0, & \\ (u_0, 0), & \xi < -l, \end{cases}$$

where

$$D = D(u_0) = \frac{q_0}{u_o} + \frac{1}{2} u_0 \tag{6}$$

$$u_0^2 \geqslant 2q_0. \tag{7}$$

The structure of the travelling wave solution is a shock wave followed by a finite reaction zone and a constant region.

For the Riemann initial data

$$u_0(x) = \begin{cases} u_0, & x \leqslant 0, \\ 0, & x > 0, \end{cases} \tag{8}$$

where u_0 satisfies (7), we have the following results.

THEOREM 2.1. *The solution to the Riemann problem* (1), (2), (8), *and* (5) *exists globally and satisfies*

$$0 \leqslant u \leqslant M, \qquad 0 \leqslant u_x \leqslant M, \qquad |u_t| \leqslant M, \qquad u_{xx} \geqslant 0,$$
$$0 \leqslant z \leqslant 1, \qquad 0 \leqslant z_x \leqslant K, \qquad |z_t| \leqslant M,$$

where M depends only on q_0, K, u_0. All the estimates involving derivatives are valid away from the shock curve $(s(t), t)$.

Furthermore, $s(t)$ satisfies

$$s'' \geqslant 0$$

and

$$\lim_{t \to +\infty} s'(t)$$

exists.

THEOREM 2.2. *The solution of* (1), (2), (8), *and* (5) *converges uniformly to the travelling wave solution and the shock front is asymptotically linear,*

$$\lim_{t \to +\infty} \sup_{x - Dt + x_0 \leqslant 0} |(u, z)(x, t) - (\psi, z)(x - Dt + x_0)| = 0$$

$$\lim_{t \to +\infty} (u, z)(x, t) = (0, 1) \qquad \textit{for} \quad x - Dt + x_0 > 0,$$

and

$$\lim_{t \to +\infty} (s(t) - Dt + x_0) = 0,$$

where

$$\int_{-\infty}^{+\infty} [u(x, 0) - \psi(x)]\, dx = -u_0 x_0.$$

Remark. We also have convergence in L^p, $p \geqslant 1$ norm. For the proof, we refer the reader to [5].

We have a comparison principle for solutions of (1) and (2) as in [5]. Since it will be used in the remaining sections repeatedly, we state the theorem here.

In the next theorem the initial value has the form

$$u(x, 0) = \begin{cases} u_0(x) & x \leqslant s(0), \\ 0, & x > s(0), \end{cases}$$

where $u_0(x)$ is a nondecreasing function.

THEOREM 2.3 (A Comparison Principle). *Suppose $u_1(x, t)$ and $u_2(x, t)$ are solutions of* (1) *and* (2) *with nondecreasing initial data $u_{10}(x)$ and $u_{20}(x)$ and shock wave positions $s_1(t)$ and $s_2(t)$, respectively. If*

$$s_1(0) < s_2(0)$$

and

$$u_{10}(x) \geqslant u_{20}(x), \qquad x \leqslant s_1(0),$$

then there is some $T > 0$, *such that if* $0 < t < T$, *we have*

$$u_1(x, t) > u_2(x, t), \qquad x \leqslant s_1(t).$$

3. A Special Solution

There is a special solution of (1) and (2) with compact support and it travels at a constant speed. In fact, it is a CJ travelling wave followed by a rarefaction wave,

$$(U, Z)(x, t) = \begin{cases} (0, 1), & x - D_{CJ} t \geqslant l, \\ (\psi, z)_{CJ}\,(x - D_{CJ} t - l), & 0 \leqslant x - D_{CJ} t < l, \\ \left(\dfrac{x}{t}, 0\right), & x - D_{CJ} t < 0, \end{cases}$$

where $D_{CJ} = \sqrt{2q_0}$ is the speed of the CJ wave which is minimum among all travelling waves. In this case, the head of the rarefaction wave moves at precisely the same speed of the front because of the sonic property of the CJ wave.

In this section, we take initial data

$$u(x, 0) = u_0(x) = \begin{cases} \psi(x - l), & 0 \leqslant x \leqslant l, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where $\psi(x)$ is a travelling wave solution satisfying

$$\psi(-l) = u_0 > \sqrt{2q_0}.$$

Hence, $D > \sqrt{2q_0}$ by (6).

The reason for considering the special initial value is that in this case we can get concaveness of the front and from which we can get asymptotic behavior of the front.

Our main results in this section are the following.

Theorem 3.1. *The solution to problem* (1), (2), (9), *and* (5) *exists globally and satisfies*

$$0 \leqslant u \leqslant M, \qquad 0 \leqslant u_x \leqslant M, \qquad |u_t| \leqslant M, \qquad u_{xx} \geqslant 0,$$

$$0 \leqslant z \leqslant 1, \qquad 0 \leqslant z_x \leqslant K, \qquad |z_t| \leqslant M.$$

Furthermore, the shock front $s(t)$ satisfies

$$\sqrt{2q_0} \leqslant s' \leqslant D,$$
$$s'' \leqslant 0, \qquad t > \tau,$$

where $\tau > 0$ is some constant.

THEOREM 3.2. *The solution converges to the CJ travelling wave solution followed by a rarefaction wave as $t \to +\infty$,*

$$\lim_{t \to +\infty} \sup_{x - (D_{CJ} t + l + x_0) \leqslant \sigma} |(u, z)(x, t) - (U, Z)(x, t + t_0)| = 0,$$

where σ is any negative number, the shift x_0 is determined from the initial data

$$\int_0^l [u_0(x) - \psi_{CJ}(x - l)]\, dx = \tfrac{1}{2} x_0 \sqrt{2q_0}$$

and $t_0 = x_0/\sqrt{2q_0}$.

Furthermore, the shock front is asymptotically linear,

$$\lim_{t \to +\infty} (s(t) - D_{CJ} t + x_0) = 0.$$

Remark. We also have the convergence result in L^p, $p \geqslant 1$ norm. This is a consequence of the conservation law and the above result.

We follow the framework of the Riemann problem in [5].

(i) Find out a function space E where the fixed point theorem base upon.

(ii) Construct an iteration and define a mapping A such that it maps E to itself.

(iii) Prove that the mapping has a fixed point. This is achieved by doing *a priori* estimates on solutions.

(iv) Use the fixed point to construct a solution and prove that the solution converges to the special solution (U, Z) as $t \to +\infty$.

3.1. *Existence*

We will prove that the solution to problem (1), (2), (9), and (5) exists for all time $t > 0$.

First we write equation (1) in characteristic form

$$\frac{dx(t)}{dt} = u(x(t), t)$$

$$\frac{du}{dt} = q_0 z_x.$$

If we denote the shock wave position by $x = s(t)$, then by the Rankine–Hugoniot condition

$$\frac{ds(t)}{dt} = \frac{1}{2} u(s(t), t).$$

From the initial data (9), we have

$$s(0) = l.$$

Solving z from Eq. (2), we have

$$z(x, t) = \begin{cases} 0, & x \leqslant s(t) - l, \\ (K(1-\alpha)(x - s(t)) + 1)^{1/(1-\alpha)}, & s(t) - l \leqslant x \leqslant s(t), \\ 1, & x > s(t), \end{cases}$$
$$=: g(x - s(t)). \tag{10}$$

Hence,

$$\frac{du}{dt} = \begin{cases} q_0 K(K(1-\alpha)(x - s(t)) + 1)^{\alpha/(1-\alpha)}, & s(t) - l \leqslant x \leqslant s(t), \\ 0, & \text{otherwise}, \end{cases}$$
$$=: f(x - s(t)). \tag{11}$$

Clearly, $f \geqslant 0$ and is continuous and nondecreasing in $(-\infty, 0)$.

From our initial data, we have

$$s(t) = Dt + l, \qquad \text{for} \quad 0 \leqslant t \leqslant \tau,$$

where $\tau > 0$ is the time at which the characteristic

$$\frac{dx(t)}{dt} = u(x(t), t), \qquad x(0) = 0, \qquad u(x(0), 0) = u_0$$

intersects $x = s(t)$.

Let

$$E=\{j: j\in C^1[0,T],\ j(t)=s(t), 0\leqslant t\leqslant \tau, D_{\mathrm{CJ}}<j'(t)\leqslant D, j\ concave\}.$$

Clearly, E is a closed bounded subset of $C^1[0, T]$.
Consider the auxiliarly problem

$$u_t+uu_x=\begin{cases} q_0K(K(1-\alpha)(x-j(t))+1)^{\alpha/(1-\alpha)}, & j(t)-l\leqslant x, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

$$=F(x-j(t)) \tag{13}$$

$$u(x,\tau)=\begin{cases} 0, & x<D\tau, \\ \psi(x-D\tau-l), & D\tau\leqslant x\leqslant D\tau+l, \\ \psi(0), & x>D\tau+l, \end{cases} \tag{14}$$

where $j\in E$.

LEMMA 3.3. *There exists a unique continuous, piecewise smooth solution of* (13) *and* (14) *for all* $t>0$ *and satisfies*

$$u_x(x,t)\geqslant 0 \qquad \textit{for} \quad t>\tau \tag{15}$$

in the sense that $u_x\geqslant 0$ *or its one-sided limits* $\geqslant 0$ *and*

$$\frac{d}{dt}u(j(t)+c,t)\leqslant 0 \qquad \textit{for} \quad t>\tau \qquad \textit{and} \qquad -l\leqslant c\leqslant 0. \tag{16}$$

Proof. Since both the initial value and the source term F are nondecreasing in x, it is easy to show that the solution u is nondecreasing.

To prove (16), we need a transformation:

$$r=t$$

$$y=x-j(t).$$

Equation (13) then becomes

$$u_r+(u-j')u_y=F(y).$$

Since u is continuous,

$$u(y=-l+,r)=u(y=-l-,r)=\frac{j(r)-l}{r}.$$

Let $p=\partial u/\partial r$.

Differentiating the above equality with respect to r, we have

$$p(y=-l+, r)=p(y=-l-, r)$$
$$=\frac{j'}{r}-\frac{j-l}{r^2}$$
$$=\frac{1}{r}\left(j'-\frac{j-l}{r}\right)<0.$$

The last inequality holds since j is concave.

Differentiating Eq. (13) with respect to r inside the reaction zone, we have

$$\frac{dp}{dr}+\frac{\partial u}{\partial y}p=\frac{\partial u}{\partial y}j'',$$

where $d/dr=\partial/\partial r+(u-j')(\partial/\partial y)$ is the derivative in the characteristic direction.

Assuming j is smooth for the moment, then $j''\leqslant 0$ by the choice of j. From (15), $\partial u/\partial y=\partial u/\partial x>0$. Applying a maximum principle in the above equation for p, we have that $p(y, r)\leqslant 0$ for $-l\leqslant y$ and $r>\tau$. In particular, it is true at $y=c$, $-l\leqslant c\leqslant 0$. That is, at $x=j(t)+c$,

$$\frac{\partial}{\partial t}u(j(t)+c, t)+j'(t)\frac{\partial}{\partial x}u(j(t)+c, t)\leqslant 0.$$

Or,

$$\frac{d}{dt}u(j(t)+c, t)\leqslant 0.$$

If j is not smooth enough, we arrive at the conclusion by approximating j by smooth functions and passing to the limit since the result only involving the first derivative of j. ▮

LEMMA 3.4. *Solution of* (13) *and* (14) *satisfies*

$$D_{\mathrm{CJ}}<\tfrac{1}{2}u(j(t), t)<D, \qquad t>0. \tag{17}$$

Proof. By definition

$$D_{\mathrm{CJ}}<j'(t)\leqslant D, \qquad t\geqslant 0.$$

Initially,

$$U(x, 0)\leqslant u_0(x)\leqslant \psi(x-l), \qquad 0\leqslant x\leqslant l.$$

Applying the comparison principle Theorem 2.3 to $U(x, t)$ and $u(x, t)$ and to $u(x, t)$ and $\psi(x - Dt - l)$, the conclusions follow immediately. ∎

The following lemma is useful in getting *a priori* estimates for the fixed point theorem.

LEMMA 3.5. *Solution of* (13) *and* (14) *is convex in* x; *that is,*

$$\frac{\partial^2 u}{\partial x^2}(x, t) \geqslant 0, \qquad \textit{for} \quad j(t) - l \leqslant x \leqslant j(t), \quad t > 0. \tag{18}$$

Proof. Since $u(x, t) = x/t$ for $0 \leqslant x \leqslant j(t) - l$, we have

$$u_x(j(t) - l-, t) = \frac{1}{t}.$$

Differentiating $u(j(t) - l, t) = (j(t) - l)/t$ along $x = j(t) - l+$, we have

$$\begin{aligned} u_t + j' u_x &= \frac{j'(t)}{t} - \frac{j(t) - l}{t^2} \\ &= \frac{1}{t}(j'(t) - u(j(t) - l, t)) < 0 \qquad \text{at} \quad x = j(t) - l+. \end{aligned}$$

Also from Eq. (13),

$$u_t + u u_x = F(x - j(t)) = F(-l) = 0, \qquad \text{at} \quad x = j(t) - l+.$$

Eliminating u_t from the above two equations, we have

$$u_x(j(t) - l+, t) = \frac{1}{t};$$

i.e., u_x is continuous across $x = j(t) - l$.

Again differentiating $u_x(j(t) - l+, t) = 1/t$ along $x = j(t) - l+$, we have

$$u_{xt} + j' u_{xx} = -\frac{1}{t^2}, \qquad \text{at} \quad x = j(t) - l+.$$

Differentiating Eq. (13) with respect to x, we get

$$\begin{aligned} u_{xt} + u_x^2 + u u_{xx} &= F'(x - j(t)) \\ &= F'(-l) = \begin{cases} 0, & \alpha > \frac{1}{2}, \\ c, & \alpha = \frac{1}{2}, \\ +\infty, & \alpha < \frac{1}{2}, \end{cases} \geqslant 0, \qquad \text{at} \quad x = j(t) - l+, \end{aligned}$$

where $c=(1/2)\,q_0 K^2$ is some positive constant. Noting that $u_x(j(t)-l+,t)=1/t$ and $u(j-l,t)-j'>0$ and eliminating u_{xt} from the above two equations, we have

$$u_{xx}(j(t)-l+,t)\geqslant 0.$$

By differentiating equation (13) with respect to x twice, we have

$$u_{xxt}+uu_{xxx}+3u_x u_{xx}=q_0 z_{xxx}.$$

Rewriting the above equation in the characteristic form, we have

$$\frac{du_{xx}}{dt}+3u_x u_{xx}=q_0 z_{xxx}\geqslant 0.$$

Initially,

$$u_{xx}(x,0)\geqslant 0, \qquad \text{for}\ -l\leqslant x\leqslant 0.$$

Applying a maximum principle along each characteristic direction, we have

$$u_{xx}(x,t)\geqslant 0, \qquad \text{for}\quad j(t)-l\leqslant x\leqslant j(t). \quad \blacksquare$$

Let s be the solution of

$$s'(t)=\tfrac{1}{2}\,u(j(t),t), \qquad s(\tau)=j(\tau).$$

From Lemma 3.3 and Lemma 3.4, we see that

$$s''=\frac{d}{dt}\,u(j(t),t)\leqslant 0,$$

$$D_{\mathrm{CJ}}<s'=\frac{1}{2}\,u(j(t),t)<D.$$

Now we are ready to define the mapping A:

$$s=Aj.$$

Clearly, A maps E into itself.

Lemma 3.6. *The mapping A has a fixed point.*

Proof. Our goal is to prove s'' bounded. In that case, $AE\subset C^{1,1}[0,T]$ is compact in $E\subset C^1[0,T]$. So there exists a fixed point by Schauder fixed point theorem.

By Lemma 3.3 and Lemma 3.4,

$$0 \leqslant u(x,t) \leqslant u(j(t),t) \leqslant 2D, \quad x \leqslant j(t);$$

i.e., u is uniformly bounded.

Next we prove that u_x is uniformly bounded. We bound $u_x(j(t)-(l/2),t)$ first. By Lemma 3.5, $u_{xx} \geqslant 0$. Hence

$$0 \leqslant u_x(j(t)-\frac{l}{2},t) \leqslant \frac{u(j(t),t)-u(j(t)-(l/2),t)}{l/2} \leqslant \frac{4D}{l}.$$

Let $x=x(t)$ be a characteristic line and t_2, t_3, and t_4 be the time at which the characteristic $x=x(t)$ intersects $x=j(t)-l$, $j(t)-(l/2)$ and $x=j(t)$ respectively. Noticing that $x'(t)$ increases and $j'(t)$ decreases, we have

$$\begin{aligned}
x'(t_3)-j'(t_3) &\geqslant x'(t_3)-x'(t_2)+x'(t_2)-j'(t_2)\\
&= x'(t_3)-x'(t_2)+\frac{j(t_2)-l}{t_2}-j'(t_2)\\
&\geqslant x'(t_3)-x'(t_2)\\
&= \int_{t_2}^{t_3} x''(r)\,dr\\
&= \int_{t_2}^{t_3} F(x(r)-j(r))\,dr\\
&= \int_{t_2}^{t_3} \frac{F(x(r)-j(r))}{x'(r)-j'(r)}\,d(x(r)-j(r))\\
&\geqslant \frac{\int_{-l}^{-l/2} F(y)\,dy}{x'(t_3)-j'(t_3)}\\
&\geqslant \frac{q_0 g(-l/2)}{x'(t_3)-j'(t_3)}.
\end{aligned}$$

Hence,

$$x'(t_3)-j'(t_3) \geqslant \sqrt{q_0\, g\left(-\frac{l}{2}\right)} > 0.$$

Let $a=u_x$. Differentiating Eq. (1) with repect to x, we get

$$\frac{da}{dt} \leqslant \frac{da}{dt}+a^2 = F'(x-j(t)), \; x-j(t) \geqslant -\frac{l}{2}.$$

Integrating along the characteristic line from t_3 to t_4, we have

$$\begin{aligned}
0 &\leqslant a(t_4) \leqslant a(t_3) + \int_{t_3}^{t_4} F'(x(t) - j(t))\,dt \\
&\leqslant u_x(x(t_3), t_3) + \int_{t_3}^{t_4} \frac{F'(x(t) - j(t))}{x'(t) - j'(t)}\, d(x(t) - j(t)) \\
&\leqslant u_x\left(j(t_3) - \frac{l}{2}, t_3\right) + \frac{\int_{-l/2}^{0} F'(y)\,dy}{x'(t_3) - j'(t_3)} \\
&\leqslant \frac{4D}{l} + \frac{F(0) - F(-l/2)}{\sqrt{q_0\, g(-l/2)}},
\end{aligned}$$

which is bounded uniformly with respect to $j \in E$. By convexity of u, $0 \leqslant u_x(x, t) \leqslant u_x(j(t), t) = a(t_4) \leqslant M$ for $j(t) - l \leqslant x \leqslant j(t)$, $t > 0$.

Using Eq. (13)

$$u_t = F(x - j(t)) - u u_x$$

we get a uniform bound for u_t.

Hence

$$s'' = u_t + j' u_x$$

is bounded uniformly with respect to $j \in E$.

Therefore, A has a fixed point. ∎

With this fixed point, we construct the following solution to problem (1), (2), (9), and (5):

$$u(x, t) = \begin{cases} u_j(x, t), & x \leqslant s(t), \\ 0, & x > s(t), \end{cases}$$

$$z(x, t) = \begin{cases} 0, & x \leqslant s(t) - l, \\ g(x - s(t)), & s(t) - l \leqslant x \leqslant s(t), \\ 1, & x > s(t). \end{cases}$$

Here u_j is the solution of the auxiliary problem (13) and (14). It is easy to check that the above is indeed a solution.

LEMMA 3.7. *Solutions of the above form are unique.*

Proof. This follows from that (1) is a conservation law. The proof is the same as that of Lemma 3.5 in [5]. ∎

By the above uniqueness, we can extend definition of $x=s(t)$ to $t=+\infty$. Also, from the fact that $s''\leqslant 0$ and $s'\geqslant\sqrt{2q_0}$, we have that

$$\lim_{t\to+\infty} s'(t)$$

exists.

Boundedness of the solution and its derivatives are obtained from Lemma 3.6 directly.

This completes the proof of Theorem 3.1.

3.2. *Convergence to CJ Travelling Wave Followed by a Rarefaction Wave*

In this section we prove that the solution to problem (1), (2), (9), and (5) converges to a shifted CJ travelling wave followed by a rarefaction wave.This shift is determined by its total initial mass. This technique was used by Tai-Ping Liu in the study of stability of shock waves [7]. It is used not only to identify the shift but also to obtained information which help the convergence proof. The conservation laws and the comparison principle give us global L^1 control over the solution.

Denote

$$\int_0^l \{u(x,0)-\psi_{\mathrm{CJ}}(x-l)\}\,dx \tag{19}$$

by $(1/2)\,x_0\sqrt{2q_0}$. Clearly, $x_0>0$.

We will show that the solution $(u,z)(x,t)$ converges to $(U,Z)(x,t+t_0)$ as $t\to+\infty$, where $t_0=x_0/\sqrt{2q_0}$. This means that the final wave front position is determined by its total mass.

LEMMA 3.8. *If x_0 and t_0 are determined as above, then*

$$\int_0^{+\infty} (u(x,0)-(U,Z)(x,t_0))\,dx=0. \tag{20}$$

Proof. Let us calculate

$$\begin{aligned}\int_0^{+\infty} (U(x,t_0)-U(x,0))\,dx &= \left(\int_0^{x_0}+\int_{x_0}^{x_0+l}\right) U(x,t_0)\,dx-\int_0^l U(x,0)\,dx\\ &=\int_0^{x_0} U(x,t_0)\,dx\\ &=\tfrac{1}{2}\,x_0\sqrt{2q_0}.\end{aligned}$$

Hence

$$\int_0^{+\infty} (u(x,0)-U(x,t_0))\,dx = \int_0^{+\infty} (u(x,0)-U(x,0))\,dx$$

$$+\int_0^{+\infty} (U(x,0)-U(x,t_0))\,dx$$

$$= \tfrac{1}{2}x_0\sqrt{2q_0} - \tfrac{1}{2}x_0\sqrt{2q_0} = 0. \quad \blacksquare$$

We bound $s(t)$ from above by the conservation law in the next lemma.

LEMMA 3.9. *Let $s_1 = D_{\mathrm{CJ}}t + l$, which is the shock wave position of $U(x,t)$. Then $0 \leqslant s(t) - s_1(t)$ is bounded.*

Proof. We prove it by contradiction. If $s(t) - s_1(t)$ is not bounded, then there exists $t_n \to +\infty$, such that

$$s(t_n) - s_1(t_n) > n.$$

Therefore we have

$$\int_{-\infty}^{+\infty} (u(x,t_n) - U(x,t_n))\,dx \geqslant \tfrac{1}{2}(n-l)\sqrt{2q_0}.$$

On the other hand, using the conservation law, the left-hand side on the above inequality is equal to its initial value

$$\int_{-\infty}^{+\infty} (u(x,0)-U(x,0))\,dx = \tfrac{1}{2}x_0\sqrt{2q_0}.$$

So we get a contradiction. That proves the lemma. $\blacksquare$

We prove that $x = s(t)$ is asymptotically linear.

LEMMA 3.10.

$$\lim_{t\to+\infty} (s(t) - (D_{\mathrm{CJ}}t + l + \delta)) = 0$$

$$\lim_{t\to+\infty} s'(t) = D_{\mathrm{CJ}}$$

for some $\delta > 0$.

Proof. Let $E(t) = s(t) - (D_{\mathrm{CJ}}t + l)$. Then $E(t)$ is bounded by Lemma 3.9. By Lemma 3.3,

$$E'(t) = s'(t) - D_{\mathrm{CJ}} \geqslant 0.$$

Therefore, $\lim_{t\to+\infty} E(t)$ exists. So there is a $\delta>0$, such that

$$\lim_{t\to+\infty}(s(t)-(D_{\mathrm{CJ}}t+l+\delta))=0.$$

By Theorem 3.1, $\lim_{t\to+\infty} s'(t)$ exists. Therefore,

$$\lim_{t\to+\infty} s'(t)=D_{\mathrm{CJ}}. \quad \blacksquare$$

Next, we prove that inside the reaction zone the solution converges to a function of $x-(D_{\mathrm{CJ}}t+l+\delta)$ as $t\to+\infty$.

LEMMA 3.11. *There are Lipschitz functions $u_\infty(\xi)$ and $z_\infty(\xi)$ such that*

$$\lim_{t\to+\infty}\ \sup_{-l\leqslant x-(D_{\mathrm{CJ}}t+l+\delta)\leqslant\sigma}|(u,z)(x,t)-(u_\infty,z_\infty)(x-(D_{\mathrm{CJ}}t+l+\delta))|=0,$$

where $\sigma<0$ is any negative number.

Proof. From (10) and (11), u and z are functions of $x-s(t)$. Using the above lemma, $s(t)$ is asymptotically linear. Hence it is easy to prove that u and z are functions of $x-(D_{\mathrm{CJ}}t+l+\delta)$ as $t\to+\infty$. The proof is the same as Lemma 4.5 in [5], except that now $x=s(t)$ approaches $x=D_{\mathrm{CJ}}t+l+\delta$ from left instead of right. $\blacksquare$

LEMMA 3.12. $(u_\infty,z_\infty)(-l)=(\sqrt{2q_0},0)$.

Proof. Following the proof of Lemma 3.4, we have

$$u(D_{\mathrm{CJ}}t+\delta,t)\geqslant u(s(t)-l,t)\geqslant\psi_{\mathrm{CJ}}(-l)=\sqrt{2q_0}.$$

Hence

$$u_\infty(-l)\geqslant\sqrt{2q_0}.$$

If $u_\infty(-l)-\sqrt{2q_0}=\gamma>0$, we will derive a contradiction. Since $u(D_{\mathrm{CJ}}t+\delta,t)\to u_\infty(-l)$ and $s'(t)\to\sqrt{2q_0}$ as $t\to+\infty$, there is a $T>0$, such that if $t>T$

$$u(D_{\mathrm{CJ}}t+\delta,t)>\sqrt{2q_0}+\tfrac{3}{4}\gamma \tag{21}$$

and

$$s'(t)<\sqrt{2q_0}+\tfrac{1}{4}\gamma.$$

In finite time $T_1 > T$, the characteristic line $dx/dt = \sqrt{2q_0} + (1/2)\gamma$ will intersect $x = s(t) - l$ so that

$$u(s(t) - l, t) = \frac{dx}{dt} \leqslant \sqrt{2q_0} + \frac{1}{2}\gamma, \qquad \text{for} \quad t > T_1.$$

Since $s(t) - l - (D_{CJ} t + \delta) \to 0$ as $t \to +\infty$ and u is continuous, we can find a $T_2 > T_1$ such that if $t > T_2$

$$u(D_{CJ} t + \delta, t) \leqslant \sqrt{2q_0} + \tfrac{5}{8}\gamma,$$

which contradicts (21). Hence

$$u_\infty(-l) = \sqrt{2q_0}.$$

By a similar argument, we have

$$z_\infty(-l) = 0. \quad \blacksquare$$

For $x - (D_{CJ} t + l + \delta) \geqslant 0$, we have $x - s(t) \geqslant 0$ and hence

$$(u, z)(x, t) = (0, 1).$$

Now define

$$(u_\infty, z_\infty)(x - (D_{CJ} t + l + \delta)) = (0, 1), \qquad \text{for} \quad x - (D_{CJ} t + l + \delta) \geqslant 0.$$

LEMMA 3.13. *$(u_\infty, z_\infty)(\xi)$ defined above is the* CJ *travelling wave solution inside the reaction zone, i.e.,*

$$(u_\infty, z_\infty)(\xi) = (\psi, z)_{CJ}(\xi), \qquad \textit{for} \quad -l \leqslant \xi = x - (D_{CJ} t + l + \delta) \leqslant 0.$$

Proof. Since (u_∞, z_∞) solves the same differential equations and satisfies the same boundary conditions as $(\psi, z)_{CJ}$, they are the same by uniqueness. $\blacksquare$

LEMMA 3.14. *$\delta = x_0$. Furthermore, $(u, z)(x, t)$ converges to the steady solution $(U, Z)(x, t + t_0)$ with $t_0 = x_0/\sqrt{2q_0}$.*

Proof. First, we prove that $(u, z)(x, t)$ converges to $(U, Z)(x, t + t_\delta)$ with $t_\delta = \delta/\sqrt{2q_0}$. By Lemma 3.13,

$$0 < u(x, t) - U(x, t + t_0) \to 0, \qquad \text{for} \quad -l \leqslant x - (D_{CJ} t + \delta + l) \leqslant \sigma < 0,$$

where σ is any negative number.

Since $s(t) - (D_{CJ} t + \delta + l) \to 0$ as $t \to +\infty$, we have

$$0 < u(x, t) - U(x, t + t_0) \to 0$$

for $s(t)-D_{CJ}t+\delta+l)-l \leqslant x-(D_{CJ}t+\delta+l) \leqslant -l$. Hence, for any $0 < x < s(t)-l$,

$$0 < u(x, t) - U(x, t+t_0) < u(s(t)-l, t) \\ - U(s(t)-l, t+t_0) \to 0, \qquad \text{as} \quad t \to +\infty.$$

That proves the convergence.

On the other hand, from Lemma 3.8, we have

$$\int_0^{+\infty} [U(x, t_0) - u(x, 0)]\, dx = 0.$$

By the conservation law,

$$\int_0^{+\infty} (U(x, t+t_0) - u(x, t))\, dx = 0.$$

That is,

$$\int_0^{+\infty} (U(x, t+t_0) - U(x, t+t_\delta))\, dx = 0.$$

Hence, we have $t_\delta = t_0$ and hence $\delta = x_0$. ▮

Combining the above results, Theorem 3.2 is proved.

4. The Initiation Problem

We consider the model (1) and (2) with the initial data

$$u(x, 0) = \begin{cases} u_0, & 0 < x < d, \\ 0, & \text{elsewhere}, \end{cases} \tag{22}$$

where $u_0 > 0$ and $d > 0$.

There are two main wave propagation mechanisms. One is wave attenuation caused by the nonlinear spreading of rarefaction waves, while the other is wave amplification due to exothermic heat release. When both mechanisms complete, the possible outcome either the sustained propagation of a wave travelling at CJ speed or a gradual decay.

First, we look at the small initial data, i.e. both u_0 and length of the support d are small. We prove that the solution tends to zero as $t \to +\infty$.

THEOREM 4.1. *If initial condition* (22) *satisfies*

$$u_0 < 2u_i,$$
$$0 < d < \frac{(u_i - (1/2)u_0)^2}{2q_0 K},$$

then the solution to problem (1), (2), (5), *and* (22) *converges to zero as* $t \to +\infty$ *at the rate* $O(1/\sqrt{t})$.

$$\lim_{t \to +\infty} \sup_x |u(x, t)| = 0.$$

Remark. The above conclusion holds for all $0 < \alpha \leqslant 1$.

Proof. The existence part is the same as the previous section.

To prove the convergence, our strategy is to prove that the straight characteristic line $x = u_i t$ inside the rarefaction wave will catch up with the shock front $x = s(t)$ in finite time. In this case, the solution will decay as that of Burger equation.

Note that

$$\begin{aligned} s'' &= \frac{1}{2}\frac{d}{dt} u(s(t), t) \\ &= \frac{1}{2}(u_t + s'(t)\, u_x) \\ &= \frac{1}{2}\left(u_t + \frac{1}{2} u u_x\right) \\ &= \frac{1}{2} q_0 K - \frac{1}{4} u u_x \\ &\leqslant \frac{1}{2} q_0 K. \end{aligned}$$

Let $\beta = u_0/2u_i$. Then $0 < \beta < 1$. Let $\delta = (u_i - (\frac{1}{2})u_0)/q_0 K$. Clearly, $\delta > 0$ since $s'(0) = \frac{1}{2}u_0 = \beta u_i < u_i$. If $0 < t < \delta$, then

$$\begin{aligned} s'(t) &= s'(0) + s''(\xi)\, t \\ &\leqslant \beta u_i + \tfrac{1}{2} q_0 K \delta \\ &= \tfrac{1}{2}(1 + \beta)\, u_i \\ &< u_i. \end{aligned}$$

Letting $x_2(t) = (1/2)(1+\beta)\, u_i t + d$, we see that

$$s(t) \leqslant x_2(t), \qquad \text{for} \quad 0 < t < \delta.$$

On the other hand, $x_1(t) = u_i t$ and $x_2(t)$ intersect at

$$T = \frac{d}{\frac{1}{2}(u_i - \frac{1}{2}u_0)} < \delta.$$

So $x_1(t)$ and $s(t)$ intersect at finite time $T_1 \leqslant T$. The solution becomes an *N-wave* after T_1.

If $t > T_1$, then characteristic lines

$$\frac{dx}{dt} = u, \qquad x(0) = 0, \qquad u(x(0), 0) \leqslant u_i$$

begin to catch up with $x = s(t)$ and hence

$$s'(t) = \frac{1}{2}\, u(s(t), t) = \frac{s(t)}{2t}.$$

Solving this ordinary differential equation for s, we have

$$s(t) = c\sqrt{t}, \qquad \text{for} \quad t > T_1$$

where $c = s(T_1)/\sqrt{T_1}$, is some constant. Therefore,

$$u(s(t), t) = \frac{s(t)}{t} = \frac{c}{\sqrt{t}}, \qquad \text{for} \quad t > T_1.$$

Finally, since $u_x \geqslant 0$,

$$0 \leqslant u(x, t) \leqslant \frac{c}{\sqrt{t}} \to 0, \qquad t \to +\infty. \quad \blacksquare$$

For initial data not small, under conditions

$$u_0 \geqslant 2\sqrt{2q_0}, \qquad d = l, \tag{23}$$

or

$$u > \sqrt{2q_0}, \qquad d \gg l, \tag{24}$$

we have the following results.

THEOREM 4.2. *Under the conditions* (23) *or* (24), *the solution to problem* (1), (2), (22), *and* (5) *exists globally and satisfies*

$$\lim_{t \to +\infty} \sup_{x - (D_{CJ}t + l + x_0) \leqslant \sigma} |(u, z)(x, t) - (U, Z)(x, t + t_0))| = 0,$$

where $\sigma < 0$ *is any negative number and the shift* x_0 *is determined from the initial data*

$$\int_0^{+\infty} [u_0(x) - \psi_{CJ}(x - l)]\, dx = \tfrac{1}{2} x_0 \sqrt{2q_0}$$

and $t_0 = x_0/\sqrt{2q_0}$.

Proof. The existence part is the same as in the previous section.

The convergence follow from the comparison principle Theorem 2.3 and results of the previous section.

Under condition (23), there is a travelling wave $\psi(x - Dt - l)$ with speed $D = D(u_0)$ such that

$$\psi_{CJ}(x - l) \leqslant u(x, 0) \leqslant \psi(x - l), \qquad \text{for} \quad 0 \leqslant x \leqslant l.$$

Let $(u_1, z_1)(x, t)$ be the solution to problem (1) and (2) with initial data

$$u_1(x, 0) = \begin{cases} \psi(x - l), & -l \leqslant x - l \leqslant 0, \\ 0, & \text{otherwise}. \end{cases}$$

We know the previous section that $(u_1, z_1)(x, t)$ converges to $(U, Z)(x, t + t_1)$ as $t \to +\infty$.

Initially, we have

$$U(x, 0) \leqslant u(x, 0) \leqslant u_1(x, 0), \qquad \text{for} \quad 0 \leqslant x \leqslant l$$

and

$$D_{CJ} \leqslant s'(0) \leqslant s_1'(0).$$

Using the comparison principle Theorem 2.3, we can prove that for all $t > 0$,

$$\begin{aligned} U(x + D_{CJ}t - l, t) &\leqslant u(x + s(t) - l, t) \\ &\leqslant u_1(x + s_1(t) - l, t), \qquad \text{for} \quad 0 \leqslant x \leqslant l \end{aligned}$$

and

$$D_{CJ} \leqslant s'(t) \leqslant s_1'(t).$$

Letting $t \to +\infty$, we have

$$\lim_{t \to +\infty} s'(t) = D_{CJ}.$$

Furthermore,

$$\lim_{t \to +\infty} (s(t) - D_{CJ} t + \delta) = 0.$$

Hence $u(x + s(t) - l, t)$ converges to the travelling wave $\Psi_{CJ}(x - l)$, for $0 \leqslant x \leqslant l$. It follows that $u(x, t)$ converges to $U(x, t + t_0)$ with t_0 determined from the initial data by the conservation law.

We now prove the conclusion under condition (24). From Theorem 2.2, we know that solution to the Riemann problem converges to $\psi(x - Dt + \delta)$ as $t \to +\infty$. For any $\varepsilon > 0$, there is a $\tau > 0$ and hence there is a d, such that

$$0 \leqslant \psi(x - s(\tau)) - u(x, \tau) < \varepsilon, \qquad \text{where} \quad -l \leqslant x - s(\tau) \leqslant 0.$$

For ε small enough, we have

$$\psi_{CJ}(x - s(\tau)) \leqslant u(x, \tau) \leqslant \psi(x - s(\tau)), \qquad \text{for} \quad -l \leqslant x - s(\tau) \leqslant 0.$$

Let $(u_2, z_2)(x, t)$ be the solution to problem (1) and (2) with initial data

$$u_2(x, \tau) = \begin{cases} \psi(x - s(\tau)), & -l \leqslant x - s(\tau) \leqslant 0, \\ \dfrac{u_0}{s(\tau) - l} x, & -s(\tau) \leqslant x - s(\tau) \leqslant -l, \\ 0, & \text{otherwise}. \end{cases}$$

It follows immediately from the last section that $(u_2, z_2)(x, t)$ converges to $(U, Z)(x, t + t_2)$ as $t \to +\infty$.

Now $u(x, \tau)$ satisfies

$$\psi_{CJ}(x - s(\tau)) \leqslant u(x, \tau) \leqslant u_2(x, \tau), \qquad \text{for} \quad -l \leqslant x - s(\tau) \leqslant 0.$$

The conclusion follows as in the last step. ∎

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REFERENCES

1. A. BOURLIOUX, "Analysis of Numerical Methods for a Simplified Detonation model," Ph.D. Thesis, Princeton University, Princeton, NJ, 1990.
2. W. FICKETT AND W. C. DAVIS, "Detonation," Univ. of California Press, Berkeley, 1979.
3. A. KOLMOGOROFF, I. PETROVSKY, AND N. PISCOUNOFF, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application a un problème biologique, *Bull. Univ. Moskov Ser. Internat. Sect. A* **1**, No. 6 (1937), 1–25.
4. P. D. LAX, Hyperbolic systems of conservation laws, II., *Comm. Pure Appl. Math.* **10**, (1957), 537–566.
5. T. LI, On the Riemann problem for a combustion model, *SIAM J. Math. Anal.* **24**, No. 1 (1993), 59–75.
6. T. P. LIU, Decay to *N-wave* of solutions of general systems of nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.* **30** (1977), 588–610.
7. T. P. LIU, "Nonlinear Stability of Shock waves for Viscous Conservation Laws," Memoris of the American Mathematical Society, No. 328, Amer. Math. Soc., Providence, RI, 1985.
8. A. MAJDA, "High Mach Number Combustion," Lectures in Applied Mathematics, Vol. 24, Amer. Math. Soc., Providence, RI, 1986.
9. R. ROSALES AND A. MAJDA, Weakly nonlinear detonation waves, *SIAM J. Appl. Math.* **43**, No. 5 (1983), 1086–1118.
10. V. ROYTBURD, private communication.
11. J. SMOLLER, Shock Wave and Reaction–Diffusion Equation," Springer-Verlag, Berlin/New York, 1983.
12. L. A. YING AND Z. H. TENG, Riemann problems for a reacting and convection hyperbolic system, *J. Approx. Theory Appl.* **1** (1984), 95–122.